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Determination of Stable and Unstable manifolds of Periodic Points of a Two Dimensional Nonlinear Map

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Abstract: In this paper we have presented a numerical technique to determine the stable and unstable manifolds of periodic points of a two dimensional nonlinear map (Henon Map) and used the Mathematica software to visualize these manifolds. Our investigation has suggested that the stable manifold at an unstable periodic point forms the boundary of a region in which the domain of attraction lies. Further, we observed the existence of homoclinic and heteroclinic points for different values of the parameters.

1. Introduction :

Invariant manifolds are important in many application areas. In the context of dynamical systems theory, stable and unstable manifolds are fundamental geometric structures. They partition phase spaces into sets of points with the same forward and backward limit sets. Stable and unstable manifolds play a major role in global bifurcation.

Bifurcation theory attempts to provide a systematic classification of the sudden changes in the qualitative behaviour of dynamical systems. A bifurcation occurs when a small smooth change made to the parameter values of a system causes a sudden qualitative change in its behaviour. So, in order to understand the various types of qualitative behaviour that are exhibited by a system, it is necessary to describe the various bifurcations that occur in the system of maps or differential equations and to determine the parameter values, called bifurcation values, at which these bifurcations occur.

Bifurcation theory is divided into two parts. The first part of the theory which is termed as local bifurcation are those in which fixed points or limit cycles (which can be treated as fixed points of Poincare map) appear, disappear, or change their stability. The change in stability is signaled by a change in one or more of the characteristic exponents of the jacobian matrix associated with that fixed point/periodic point. In general the characteristic exponents can be complex numbers. In case of continuous dynamical systems, at a local bifurcation, the real part becomes equal to 0 as some parameter (or parameters) of the system is changed. As the real part of the characteristic exponent changes from negative to positive, the motion associated with that characteristic direction goes from being stable (attracted toward the fixed point) to being unstable (being repelled by the fixed point). For a fixed point/periodic point of maps, this criterion is equivalent to having the absolute value of the characteristic multiplier equal to unity.

The other part of the theory, known as global bifurcations are bifurcation events that involve changes in basins of attraction, appearance of homoclinic or heteroclinic points and orbits, or other structures that extend over significant regions of state space. Since we need to take into account behaviour over a wide range of state space, a different means of classifying and studying such bifurcations is obviously needed. Global bifurcation cannot be detected through the analysis of the eigenvalues of the jacobian matrix associated with equilibria or cycles. Theory of global bifurcations is both more difficult and less articulated than in the theory of local bifurcations.

Determination of stable and unstable manifolds and the basin of attraction is thus an important aspect in the study of global bifurcations.

The rest of the paper is organized as follows :

In section-2, we provide some definitions required for our investigation. In section -3 we have mentioned aims of our investigation. Numerical technique to find out the stable and unstable manifolds is described in section-4. Section 5 includes our conclusions.

2. Some Definitions:

2.1 Homoclinic and Heteroclinic Orbits:

Some of the theory involved in the bifurcations to chaos for flows and maps is a result of the behavior of the stable and unstable manifolds of saddle points. It is possible for the stable and unstable manifolds to approach one another and eventually intersect as a parameter varies. When this occurs, there is said to be a homoclinic (or heteroclinic) intersection and the orbit is called a homoclinic (or heteroclinic) orbit. The homoclinic as well as the heteroclinic orbits are examples of separatrix, where a separatrix means a orbit that divides the phase plane into two distinctly different types of qualitative behavior. The intersection is homoclinic if a stable/unstable branch of a saddle point crosses the unstable/stable branch of the same saddle point, and it is heteroclinic if the stable/unstable branches of one saddle point cross the unstable/stable branches of a different saddle point.

2.2 Stable and Unstable Manifolds:

We define the stable manifold of a hyperbolic periodic point $\bar{x}$ having period k as the set of points $\bar{y}$ for which $(f^k)^m(\bar{y}) \to \bar{x}$ as $m \to \infty$. Also the unstable manifold of a hyperbolic periodic point $\bar{x}$ is defined as the set of all points $\bar{z}$ for which $(f^k)^m(\bar{z}) \to \bar{x}$ as $m \to -\infty$. The stable and unstable manifolds of a periodic point $\bar{x}$ are denoted by $W^s(\bar{x})$ and $W^u(\bar{x})$ respectively.

Linear stable manifold of a critical point is a straight line along which the trajectories move towards the critical point as time tends to infinity and is determined by the eigenvector corresponding to the eigenvalue with magnitude less than one of the linearized Jacobian matrix of the nonlinear system. Similarly linear unstable manifold of a critical point is a straight line along which the trajectories move away from the critical point as time tends to infinity and is determined by the eigenvector corresponding to the eigenvalue with magnitude greater than one of the linearized Jacobian matrix of the nonlinear system.

But nonlinear stable and unstable manifolds are very complicated in nature with a well established property that these are tangents to their linear counterparts in a very small neighborhood of the critical point.

2.3 The Domain of Attraction for A Periodic Point:

Let $\bar{x}$ be an attracting periodic point of period k for a diffeomorphism f. Then the set,

$$D = \{y \in R^n : (f^k)^m(y) \to \bar{x} \text{ as } m \to \infty\}$$

is the domain of attraction for $\bar{x}$.

The basin of attraction or the domain of attraction of a critical point is actually the region in which if a trajectory starts, then it approaches the critical point as number of iterations increases, i.e. the basin of attraction is that set of initial points each of which gives rise to a trajectory that approaches the critical point as number of iterations tends to infinity.

3. Aim of our study:

Our main aim is to study the domain of attraction of a stable periodic orbit. It is already well established [1, 4, 5] that when we find period doubling bifurcations in a nonlinear map f, then

(i) There exists an infinite number of periodic solutions of different periods.
(ii) There exists an uncountably infinite set of points which exhibit random behavior when f is iterated.
(iii) There is an extreme sensitivity to initial conditions.

To describe the situation in a comprehensive manner, we consider the Henon map

$$H_{\mu,b}(x, y) = (1 - \mu x^2 + y, bx) \text{ where } \mu \text{ and } b \text{ are real parameters.}$$

It is well known that this map follows period doubling route to chaos for $-\infty < b < \infty$ and μ as the control parameter [1, 3, 5, 6].

4. Numerical Methods:

Maroto [6] has shown analytically the existence of a transversal homoclinic point for small values of b and some appropriate values of μ ($\mu > 1.55$). Curry [2] in his paper suggested the existence of such a point for b = 0.3, and $\mu = 1.4$. Later, Misiurewicz and Szewc in [7] have proved rigorously that there does exist a transversal homoclinic point for b = 0.3, $\mu = 1.4$. Now, the question arises- "Does there exist a transversal homoclinic point for a higher value of b and a smaller value of μ?" This question has an affirmative answer. We present this fact for b = 0.8, and $\mu = 0.9$. Eventually, we show the existence of heteroclinic points for these parameter values.

To present our investigation in a comprehensive manner, we consider the parameter values b = 0.8, and $\mu = 0.9$. At this value of b, the second and the third bifurcation values of μ are 0.85 and 0.964285570069 respectively. So, for these b and μ there exists a stable trajectory of period-4 comprising of the following periodic points.

P(-1.149409918717, 1.080458237198)

Q(-0.634709659927, 0.713143513662)

R(0.891429392077, -0.919527934974)

and S(1.350572796497, -0.507767727942).

Again for these b and μ, there are two unstable periodic points of period-2 as

T(-0.925264339232, 0.917989249163)

and U(1.147486561454, -0.740211471386)

Furthermore, another two unstable fixed points given by the equations

$$x = \frac{(b-1) \pm \sqrt{(1-b)^2 + 4\mu}}{2\mu}, \quad y = bx \qquad \text{are}$$

V(0.948821334908, 0.759057067926)

and W(-1.171043557130, -0.936834845704)

Now, to exhibit stable and unstable manifolds at these unstable periodic points, some numerical techniques are employed as described below:

4.1 Determination of Stable and Unstable Manifolds:

The Stable Manifold Theorem [5] states that if there exists a hyperbolic point of period k, $A(x_0)$ such that the eigenvalues of the Jacobian of the transformation H^k (k is the appropriate period), at this point are λ and γ with $|\lambda| < 1$ and $|\gamma| > 1$, then there exist stable and unstable manifolds W^s, W^u at $A(x_0)$ such that they are tangent to the eigenspaces E^λ, E^γ generated by the eigenvectors corresponding to the eigenvalues λ and γ respectively.

We now wish to illustrate the procedure of obtaining the stable and the unstable manifolds at the unstable periodic points T, U, V and W.

4.1.1 At the point T(-0.925264339232, 0.917989249163):

Consider first the unstable point T which is a periodic point of period 2. The eigenvalues of the Jacobian of H^2 (two times iteration of the Henon map) at this point are found to be $\lambda = -0.465668733237$ and $\gamma = -1.374312667662$. So by the stable manifold theorem, there exist stable and unstable manifolds at T. Here, λ corresponds to the stable manifold and γ corresponds to the unstable manifold.

The linear stable and unstable manifolds of T are found to be given respectively by the straight lines

$$y = -1.017977971764104x - 0.02390946623200925 \quad (1)$$

$$y = -5.446395567632982x - 4.121366346908131 \quad (2)$$

Consider now the line (1) that passes through the point T. We consider a closed interval on this line, centered at T, having the length 0.01 units, and then pick up 1000 equally spaced points from this interval.

It is found that the inverse map H^{-2} at each of these 1000 points can be iterated up to 5 times without overflowing, although at some points more iterations are possible. Therefore, the points obtained by iterating five times the inverse map H^{-2} at these points are plotted to obtain the stable manifold at T which is shown in the following figure.

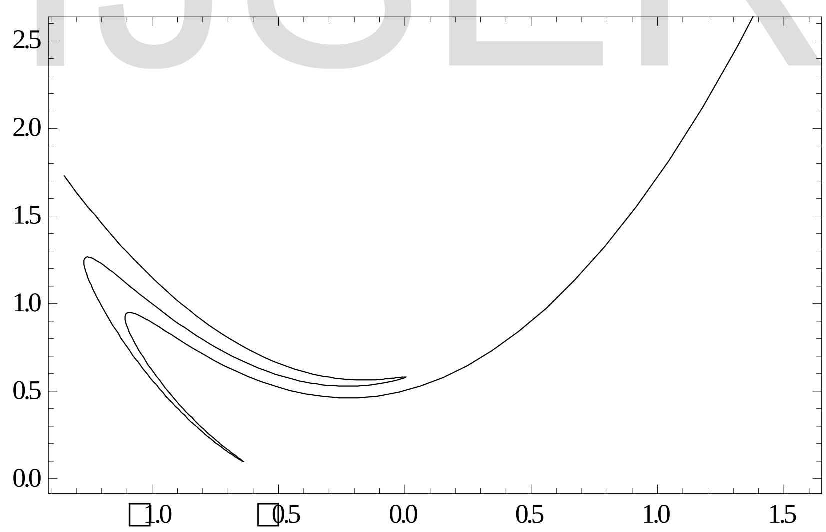

Fig 1: Stable manifold of T

Next, we consider the line (2) that also passes through T. Just like the stable manifold, we consider a segment of the line (2), centered at T, containing 1000 equally spaced points and having the total length 0.01 units. It is observed in this case that even the large number of iterations of the map H^2 does not give real overflow and that the points obtained by these iterations approach normally towards the periodic points P and Q . The number of iterations of the map H^2 we carried out is 10 and the corresponding unstable manifold is shown Fig. 2.

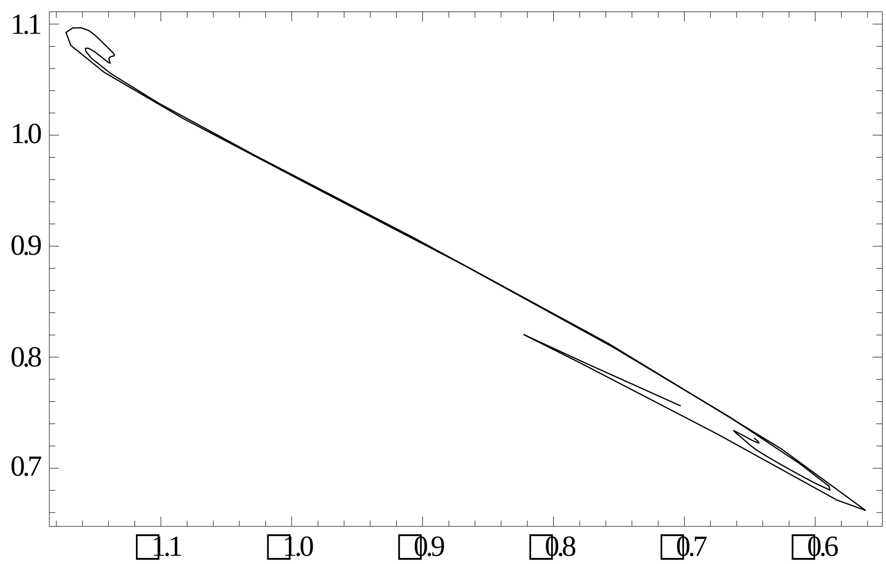

Fig 2: Unstable manifold of T

4.1.2 At the point U:

Since U is the image of T under the map H, the images of the stable manifold and the unstable manifold under H give respectively the stable manifold and the unstable manifold at the point U.

4.1.3 At the points V and W:

Following the above procedure the stable and unstable manifolds of V and W can also be determined. Since V and W are unstable periodic points of period one, so to determine the stable manifold we need to iterate the map H^{-1} and to determine the unstable manifold we need to iterate the map H. To determine the stable and unstable manifolds of V, we considered 10 iterations of H^{-1} and 6 iterations of H respectively. Further, to determine the stable and unstable manifolds of W, 4 iterations of H^{-1} and 10 iterations of H respectively were considered. The manifolds obtained by the above mentioned procedure are shown below :

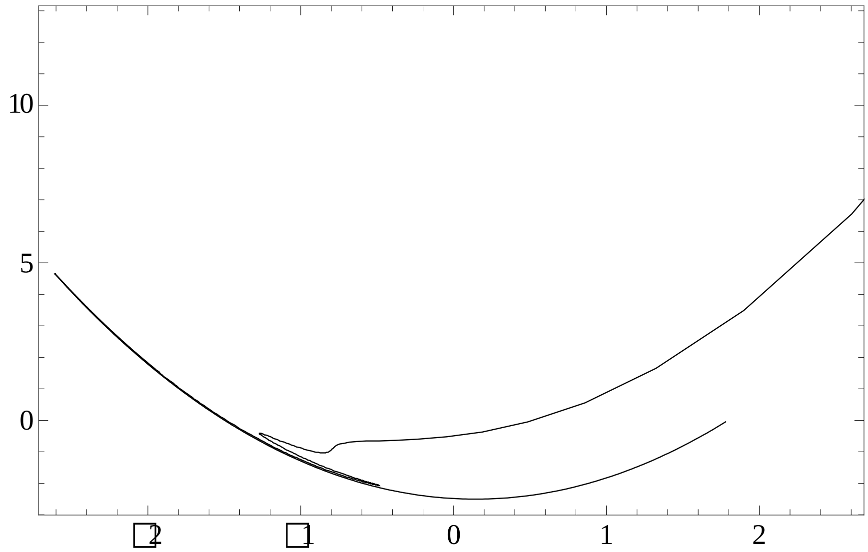

Fig 3: Stable manifold of V.

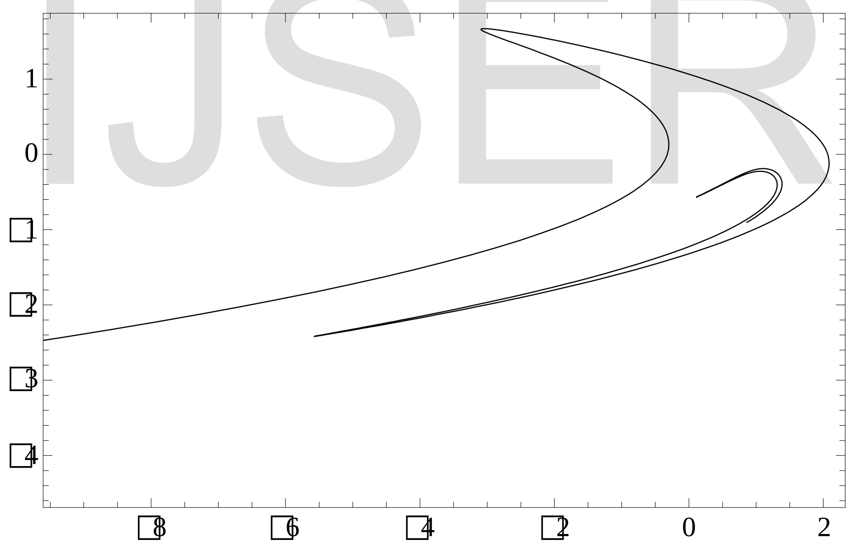

Fig 4: Unstable manifold of V.

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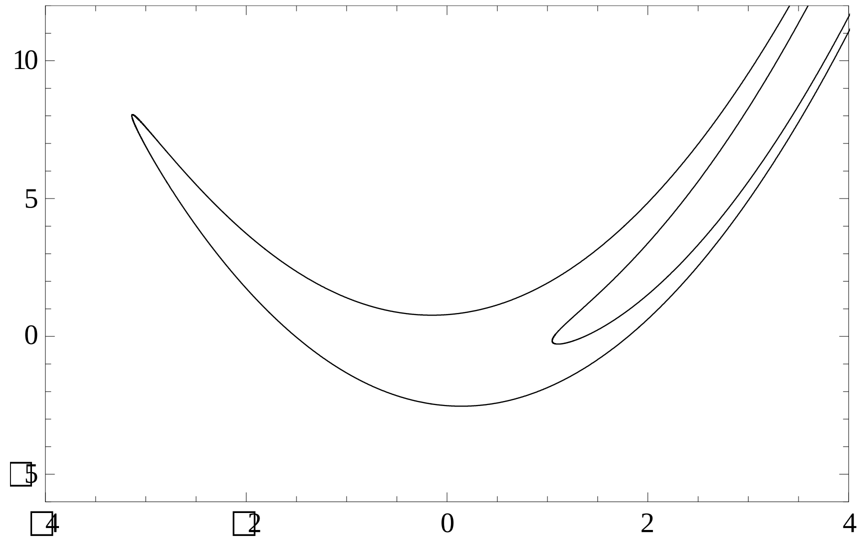

Fig 5: Stable manifold of W.

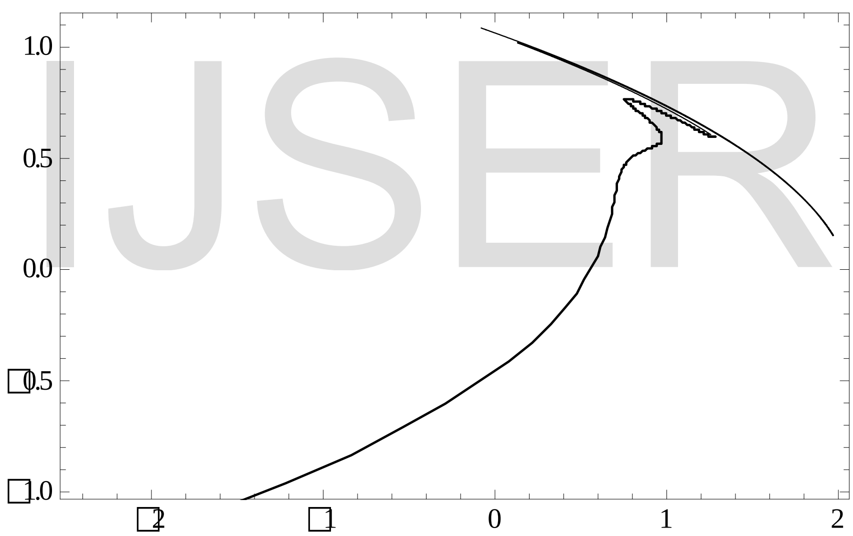

Fig 6: Unstable manifold of W.

5. Conclusions:

Conclusion 1:

The stable manifold at the unstable point W is quite interesting. It is seen from its graph (fig. 7) that all stable and the unstable points (other than W which is on the curve) are in a region whose boundary is mostly covered by this manifold. Further, it is also evident from fig. 8 that the stable manifolds at T, U and V, (which separate the domain of attraction of the periodic points P, Q, R and S), lie inside this region. So, it is important to note that this stable manifold forms mostly the boundary of a region which contains the domain of attraction of the stable periodic orbit containing the periodic points P, Q, R and S. However the violent winding of this manifold implies that the domain of attraction has a complicated boundary.

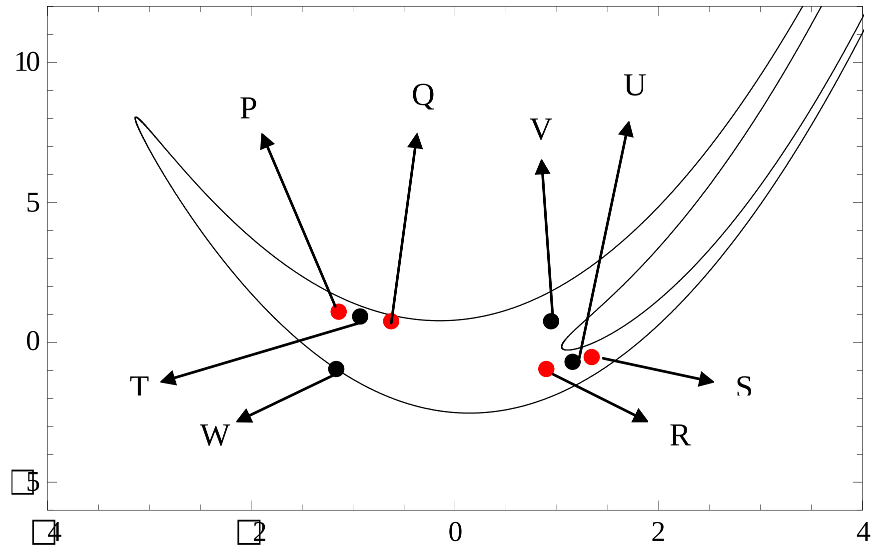

Fig 7: Stable manifold of W along with the periodic points. Red dots are the stable periodic points and black dots are the stable periodic points.

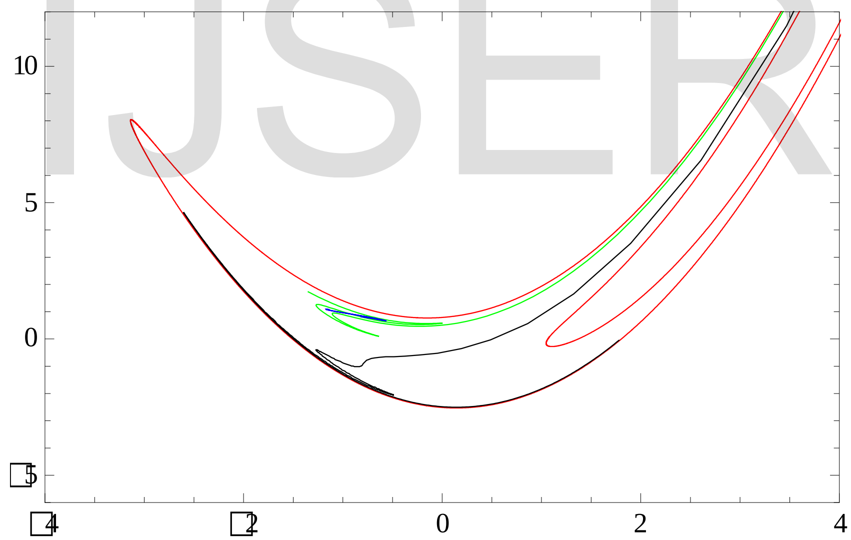

Fig 8: Stable manifolds of all the points T, U, V, W together. The stable manifolds of T (Green), U (Blue), V (Black) lies within the stable manifold of W (Red) which forms the basin boundary.

Conclusion 2 :

The unstable manifold at V intersects the stable manifold at a point other than V. Moreover, it is clear from fig.9 that this intersection is transversal. This shows the existence of a transversal homoclinic point for the parameter values b = 0.8 and $\mu = 0.9$ in the Henon map. In Fig 10, we have also shown that the unstable manifold at W intersects transversally the stable manifold of V at a point other than W and V by considering a suitable interval on the appropriate eigenspace, indicating the existence of transversal heteroclinic points.

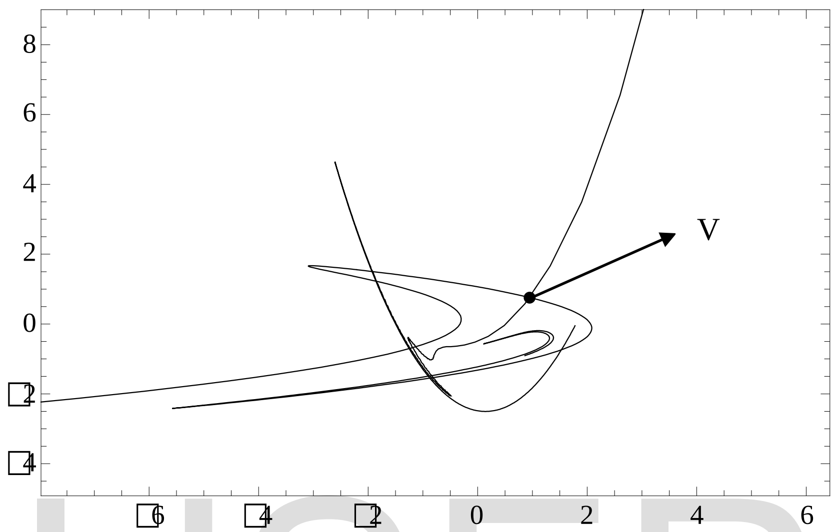

Fig 9: The stable and unstable manifolds of V intersect at points other than V indicating the existence of homoclinic points. The point V is shown by the large black dot in the figure.

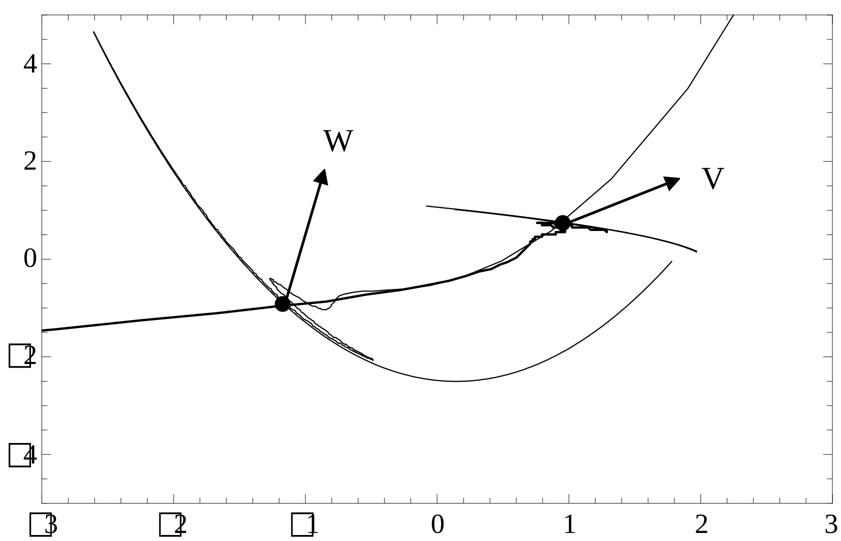

Fig 10: The unstable manifolds of W intersects the stable manifold of V at points other than W and V indicating the existence of heteroclinic points. The points W and V are shown by the large black dots in the figure.

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